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Note

Diagnosing Oscillatory Growth or Decay*

In numerical simulations one often has reason to anticipate linear behavior of the simulated system and that therefore any monitored real variable should eventually manifest a time dependence like the real part of a (generally) complex exponential function of the time.

Ignoring the case of several modes with exactly the same growth rate γ (this excludes strictly nondissipative systems with more than one purely real frequency), one would like to see four real constants emerge from the record in time steps δt, namely amplitude, phase, growth or decay rate and frequency. Four real data records are needed for the determination of these four constants.

With the notation $x + iy = e^{(\gamma + i\omega)\delta t}$ and $r = (x^2 + y^2)^{1/2}$, so that $e^{\gamma\delta t} = r$ and $\cos \omega\, \delta t = x/r$, the nth entry of the monitored real variable, f_n, is eventually given by

$$f_n = \mathrm{Re}[(a + ib)(x + iy)^n]$$

and it is desirable to combine one's last four entries of f_n in such a way as to get a running record of a, b, $\gamma\, \delta t$, and $\omega\, \delta t$. Then one can watch this record settle down to constancy. The first three equations,

$$f_0 = a,$$
$$f_1 = ax - by$$

and

$$f_2 = a(x^2 - y^2) - 2xby = 2xf_1 - r^2 f_0,$$

allow one to deduce the general recurrence relation

$$f_{n+1} = 2xf_n - r^2 f_{n-1}$$

and two successive such relations yield

$$r^2 = \frac{f_{n+2}f_n - f_{n+1}^2}{f_{n+1}f_{n-1} - f_n^2}, \tag{1}$$

$$2x = \frac{f_{n+2}f_{n-1} - f_{n+1}f_n}{f_{n+1}f_{n-1} - f_n^2}. \tag{2}$$

These are the quantities which one should monitor and which should become constant as the fastest growing, or most slowly decaying, oscillatory mode emerges.

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The constants a and b are usually of less interest. The growth rate γ and the radian frequency ω are given by

$$\gamma = \frac{1}{2\delta t} \ln r^2$$

$$\omega = \frac{1}{2\delta t} \cos^{-1}\left(2\,\frac{x^2}{r^2} - 1\right)$$

These formulas are chosen so as to eliminate the need for evaluating r as the square root of r^2.

In terms of the eigenvalues of the linear operator which advances the system by one time step, one can say that (1) gives the product of the largest pair, (2) their sum. With this interpretation, the formulas are still correct when the largest eigenvalues are real. The argument of the inverse cosine then exceeds unity, leading to hyperbolic cosines. The algorithm in this case serves to isolate the fastest growing mode from its next runner-up, while the latter still significantly affects output.

The case of zero denominators in (1) and (2), is, of course, that of "critical damping": the modes have coalesced and the output is already in geometric progression. A running record of f_n/f_{n-1} may be worth keeping in any case: r^2 and $2x$ can be generated from three successive entries of that record in an obvious manner.

The method has been used successfully in tests of the field propagator built into Stanford's 3-d electromagnetic plasma code, with the particle dynamics deliberately linearized.

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